4.6.1 On the Equilibrium between Offer and Use – a practical example from a Swiss Upper Secondary School

a) The offer-and-use model within the context of dialogic learning

Peter Gallin

Within the EU-Fibonacci Project, the Swiss TC1 (University of Zurich) has entirely focussed on inquiry-based mathematics education (IBME). To this end, dialogic learning was used as a base concept (Cf. 3rd contribution in chapter 2 (2.3): The basic patterns of Fibonacci as an overarching concept for successful implementation of IBME in international contexts). In the following, a close look at the offer-and-use model shall shed light on Peter Gallin's concept of dialogic learning, which will be underpinned by a practical example taken from our Sekundarstufe II (upper secondary level). To this end, the dialogic learning cycle (introduced in chapter 2.3) will be divided into two parts: one part is concerned with what the teacher offers, while the other relates to what use the learner makes of this offer, which is, in fact, the student's overall responsibility. This kind of view of school in general and classroom lessons in particular, in which a distinction between offer and use is drawn, originally goes back to Prof. Helmut Fend of the University of Zurich[1] and can be flawlessly embedded in the dialogic learning cycle (Diagram 1). Thus, core idea and task construction (we call them assignment) belong to the offer that the teacher makes, while keeping a journal and receiving (or giving) feedback are entirely focussed on the use, on the way how the children or students worked with the offered task. Indeed, the student will automatically perform these activities in the course of working on his tasks. The norms – i.e. the theories and rules that the curriculum expects the students to learn – form the goals of dialogic learning and should, ideally, result from the fusion of offer and use. Thus, the norms do not make the starting point but follow as consequences in later classroom units.

[1] For an explanation on the offer-and-use model see: Urs Ruf, Stefan Keller, Felix Winter (ed.): Besser lernen im Dialog, Dialogisches Lernen in der Unterrichtspraxis. Klett/Kallmeyer Verlag 2008, p. 14.

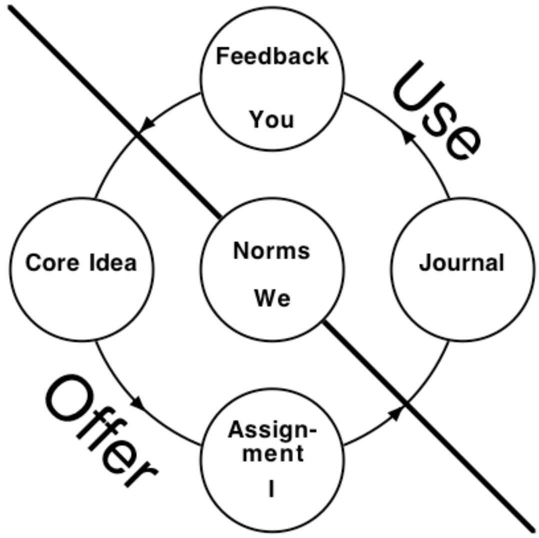

Diagram 1: The offer-and-use model within the cycle of dialogic learning

Central to this model is the fact that the quality of a classroom lesson is at its highest if the time available is evenly distributed to both parts of offer and use. This then means that about half the time should be used in conjunction with the use-part, i.e. the question how students understand and process the objects they are given to deal with. This kind of requirement strongly contrasts with how lessons are taught at most schools. In fact, this valuable approach is hardly even treated during vocational teacher training. Even within the Fibonacci project, what often seems to be at the centre of investigations is the quest only to make what the teacher offers more interesting and maybe more closely related to real-life problems. The phenomenon of putting excessive emphasis on what the teacher can do is widespread since teachers naturally ask themselves what they can do to improve the situation. However, as it is impossible to foretell what the students will make of the (improved) offer, the sight of student use is often lost at the planning stage. Moreover, the fact that student reactions cannot be planned for, and will often vary depending on the exact circumstances, is a real dilemma for innovative approaches. Then again, a hopefully successful IBME approach should not set out to assume that all results can be foretold by the teacher. If this were possible, it would invariably imply that there is no room for genuine student inquiry. As a consequence of this, inquiry-based education is only possible if the emphasis is put on the student’s use of what is offered and if this use makes the central aspect in the classroom. It goes without saying that this, in turn, presupposes an accordingly stimulating teacher offer.

Dialogic learning is consciously permeated by journal entries and through it the extent to which a student engages with the task offered is given the necessary weight. The dialogic approach is, thus, situated somewhere between instruction and (knowledge self-) construction. At the same time, it takes into account that knowledge transfer through instruction (offer) is quite effective, but that real learning is a constructive process (use) where self-motivated learning brings about truly lasting and flexible results.

The shift of emphasis towards the aspect of use simultaneously moves the burden away from the teacher: more than ever, he is now in a position to concentrate on the learning goals set in the curriculum, and does not have to plan innumerable lessons in advance.[2] In other words, the offer can be a simple one and, thus, the pressure exerted by time management issues is greatly reduced. At this point in our paper, another aspect in the discussion revolving around improvement to the classroom needs to be addressed. It is often postulated – by laymen and professionals alike – that the quality of mathematics lessons can be improved by more frequently relating the mathematical aspects in question to real-life situations. It is generally felt that only these situations can lead to successful inquiry-based work. The latest studies in this field contradict this view. In connection with Anna Susanne Steinweg's thesis "Zur Entwicklung des Zahlenmusterverständnisses bei Kindern: Epistemologisch-pädagogische Grundlegung" [On the development of childrens' comprehension of number patterns: basics in pedagogy and epistemoloqy] (University of Dortmund, Germany, 2000), the Madipedia[3] index of the institute for mathematical didactics notes:

The main result of this study is that more than half the participating children were able to recognise and describe number patterns even though they had not received prior tuition in connection with number patterns at all. When presenting the tests to these children, the author purposely refrained from setting the patterns into context or relating them to real life. The circumstance that the children worked on the task with motivation strongly hints at the fact that mathematics in it pure form is appropriate for children.

Thus, even for children at primary school level, the relation of mathematics (or its absence) to everyday life or to real-life applications is not a crucial factor when it comes to motivation. What is important, however, is that the learner is given an adequate period of time to deal with a mathematical topic and that the learner's effort to gain insight into such a topic is duly appreciated. In a school class with more than 10 people, this is only possible if each single student is given adequate space and can voice his thoughts in a learning journal.

Through their study, Deci and Ryan have shown that this procedure influences the learners' motivation the most.[4] According to the authors, the three basic pillars are "experience of autonomy", "experience of social embedding", and "experience of competence". Exactly these three experiences are central to Dialogic Learning: the self is allowed to experience autonomy through the task because it is allowed to give voice to its thoughts and feelings. Social embedding is created by feeding back authentic student texts to the whole group. Finally, it is the teacher's duty to collect

[2] Cf. article by Markus Jetzer (4.6.2)
[3] Link: „http://www.madipedia.org/index.php/Kategorie:Dissertationen_2000" (09.10.2012)
[4] Ryan, Richard M. and Deci, Edward L. (2000): Self-Determination Theory and the Facilitation of Intrinsic Motivation, Social Development, and Well-Being. In: American Psychologist. 55 (200), 68-78.

and bring together subject theory and insights developed in the student text in such a manner that the students recognise their input and thus experience competence.

Overall, inquiry-based mathematical education is not primarily connected with a given set of topics. Rather, it hinges on the way how tasks for students are formulated (offer) and what the students make of them (use). Through this approach, topics that are ordinarily part of the curriculum can be turned into dialogic learning tasks. Already after sifting through a first batch of journals, the teacher can see what insights the students have reached and what problems they have encountered. These aspects can be made essential parts of the next lesson. As a consequence, lesson planning is facilitated and evenly spread across the whole teaching time and does not require to be planned weeks ahead.

In preparation for a more demanding example, we introduce our practical example with a simple task that could not be easier. When teaching children multiplication tables, it is not uncommon to give students a question that does not lead to an investigation and that can only be answered wrongly or correctly: how much is $49 \cdot 51$? The same question can, however, easily be turned into an inquiry-based task by asking, "Show me how you calculate $49 \cdot 51$!" It becomes immediately clear that there is no single right answer and that several different approaches will lead to a fruitful class discussion on how to multiply numbers. This will be the case all the more if the students are requested to hand in their personal answers in writing through their learning journals.

b) An actual example from a mathematics course by Bruno Lustenberger

In accordance with the cycle in diagram 1, inquiry-based mathematical education will be divided into four stages, which follow the offer-and-use model. These parts together set the minimum that is necessary to show the strength of an inquiry-based working approach in the school context. The following short overview will illustrate the equilibrium between the teacher's offer and the students' use:

- The students are given a task that is closely linked to a topic set in the curriculum and that allows individual approaches. (Initial offer)
- The students keep track of their thoughts, problems and findings in their learning journal. (First use)
- The teacher organises an exchange of thoughts among the learners and gives individual feedback on remarkable insights. (Second offer)
- The teacher collects and re-distributes interesting results as well as findings that allow the group to continue the investigation. (Second use)

The following example from Bruno Lustenberger's classroom (Kantonsschule Glattal, Grammar School in Dübendorf, Switzerland) shows that this approach may, of course, lead to surprising results. His class MN5 (with a study emphasis on mathematics and physics) previously treated a number of arithmetic and geometric sequences in a traditional way. Let us now take a close look at how the four stages of inquiry-based mathematics education developed in his class in November 2011.

First offer: the task (assignment)

- Try to define and investigate at least one more type of a sequence.
- If possible, write down both the recursive and the explicit formula for your sequence(s).
- Illustrate your sequence(s) with examples.

First use: the journal of Abdullah, Ceren and Kevin

The work of Abdullah, Ceren and Kevin is used to represent the great number of contributions that are exceptionally interesting and that merit a thorough inspection. To start with, the three wrote, "Up to now, we have always added a fixed number to get to the next term. Now, we will remove the 'fixed', and if we do this, then the sequence of differences between two terms in itself becomes a sequence [an arithmetic sequence]". They continued to investigate the sequence 1, 2, 4, 7, 11, 16, 22, 29, … where the terms in the sequence of the differences are natural numbers. Because they were already familiar with the formula for the series of the first n natural numbers, they concluded that the explicit formula for their sequence should be $a_n = a_1 + \frac{n(n-1)}{2}$. After a second but not quite successful example – the sequence of perfect squares – and in order not to become lost, they turned to a sequence with greater terms and differences (Diagram 2).

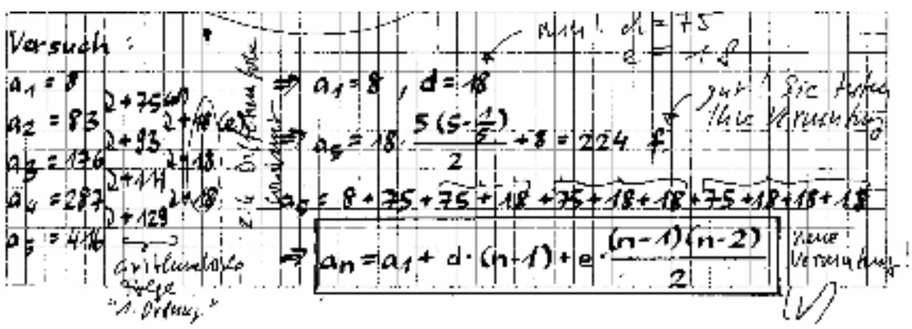

Diagram 2: Investigation of a 2nd order arithmetic sequence

Purely to increase legibility, we here provide a transcript of the students' approach:

Term	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	...
Term value	8	83	176	287	416	563	728	911	...

1. seq. of differences	75	93	111	129	147	165	183	...	
2. seq. of differences	18	18	18	18	18	18	...		

With a certain degree of agility, they investigated how $a_5 = 416$ had developed from the first term a_1, the starting number 75 in the first sequence of differences and the starting number 18 in the second sequence of differences:

$$a_5 = 416 = 8 + 75 + \overbrace{75 + 18}^{93} + \overbrace{75 + 18 + 18}^{111} + \overbrace{75 + 18 + 18 + 18}^{129}$$

The students recognised the pattern $a_5 = 8 + 75 \cdot (5-1) + 18 \cdot \dfrac{(5-1)(5-2)}{2}$, introduced the parameters $d = 75$ and $e = 18$ and concluded that

$$a_n = a_1 + d \cdot (n-1) + e \cdot \frac{(n-1)(n-2)}{2}$$

Inspired by their success, they made a forecast (albeit a wrong one to start with) for the next higher stage, i.e. a sequence where only the 3rd sequence of differences would be constant:

$$a_n = a_1 + d \cdot (n-1) + e \cdot \frac{(n-1)(n-2)}{2} + f \cdot \frac{(n-1)(n-2)(n-3)}{3}$$

Diagram 3 shows this extract from their learning journal.

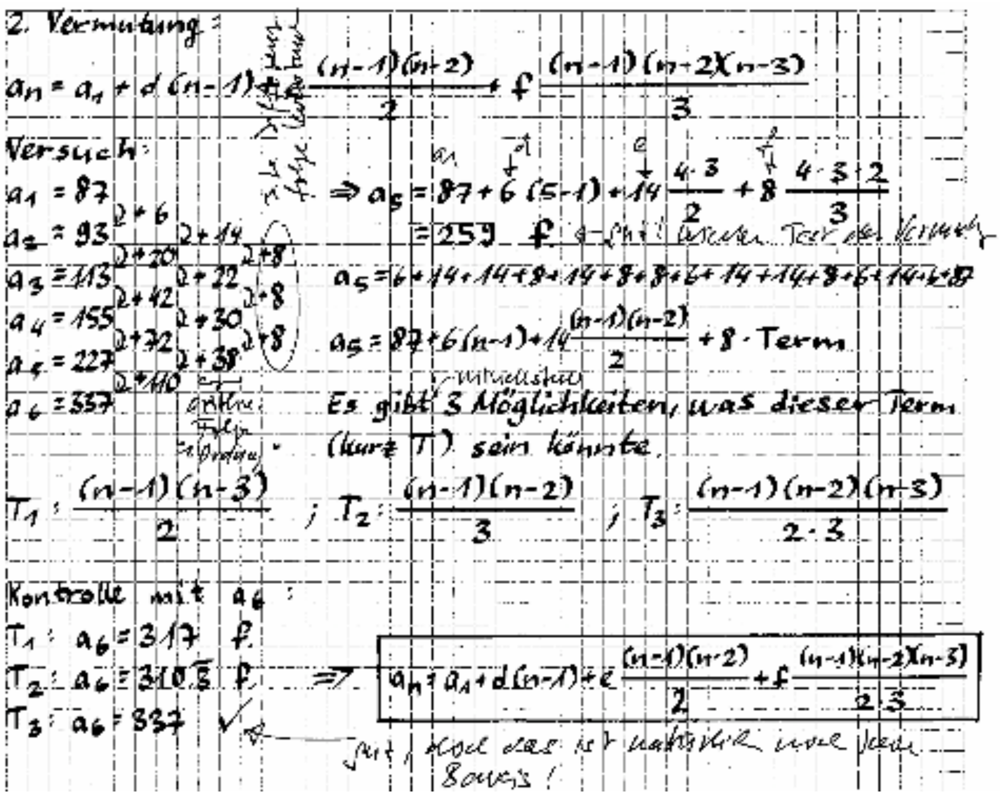

Diagram 3: Investigation of an arithmetic 3rd order sequence

For their example, the three students chose the new sequence

Term	a_1	a_2	a_3	a_4	a_5	a_6	...
Term value	87	93	113	155	227	337	...
1. seq. of differences	6	20	42	72	110	...	
2. seq. of differences	14	22	30	38	...		
3. seq. of differences	8	8	8	...			

Once again, they applied their method of back tracing to the first terms of the sequences of differences and inspected: $a_5 = 227$

$$a_5 = 227 = 87 + 6 + \overbrace{6 + 14}^{20} + \overbrace{6 + 14 + \overbrace{14 + 8}^{22}}^{42} + \overbrace{6 + 14 + \overbrace{14 + 8}^{22} + \overbrace{14 + 8 + 8}^{30}}^{72}$$

They were only unsure about the number of terms (summands) 8 and so wrote

$a_5 = 227 = 87 + 6 \cdot (5-1) + 14 \cdot \dfrac{(5-1)(5-2)}{2} + 8 \cdot \text{Term}$, where "Term" stands for one of the three expressions $T_1 = \dfrac{(n-1)(n-3)}{2}$; $T_2 = \dfrac{(n-1)(n-2)}{3}$; $T_3 = \dfrac{(n-1)(n-2)(n-3)}{2 \cdot 3}$.

Thus, they reached the conclusion that T_3 had to be the right one, and again they introduced the parameters $d = 6$, $e = 14$ and $f = 8$. At this point, they had reached a general and now correct formula

$$a_n = a_1 + d \cdot (n-1) + e \cdot \frac{(n-1)(n-2)}{2} + f \cdot \frac{(n-1)(n-2)(n-3)}{2 \cdot 3}$$

Now, there was no stopping them (diagram 4). Without knowing the name of their sequence, they successfully put their formula to the test with an arithmetic 4th order sequence.

$$a_n = a_1 + d \cdot (n-1) + e \cdot \frac{(n-1)(n-2)}{2} + f \cdot \frac{(n-1)(n-2)(n-3)}{2 \cdot 3} + g \cdot \frac{(n-1)(n-2)(n-3)(n-4)}{2 \cdot 3 \cdot 4}$$

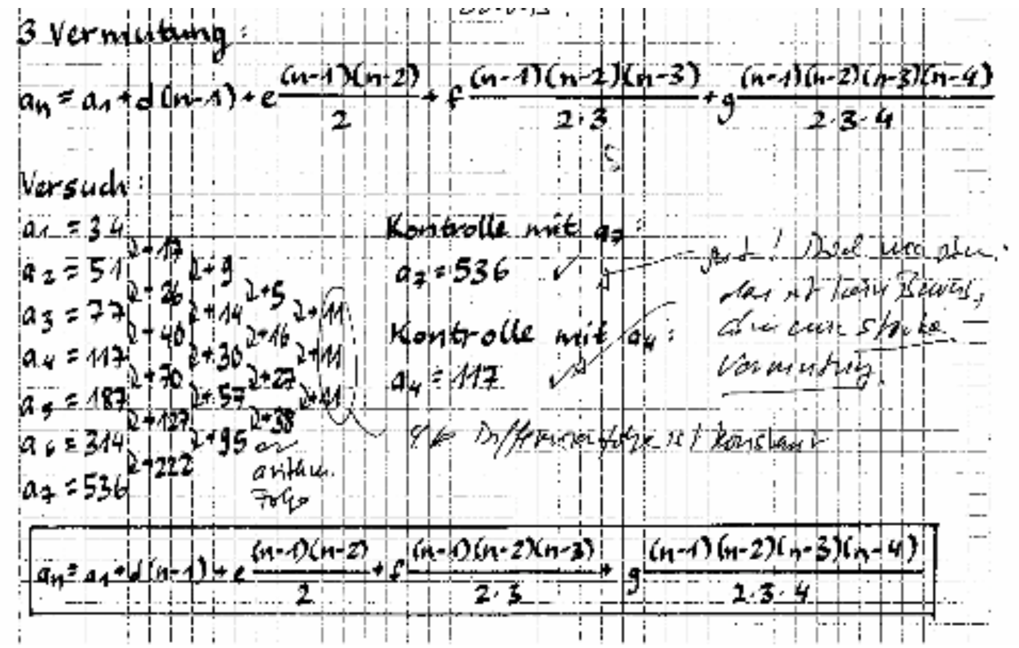

Diagram 4: Investigating an arithmetic 4th order sequence

Second offer: guided reflection and proof by teacher

In the course of their investigation on “self-made sequences”, the three students developed a building principle for arithmetic sequences of k - th order, which is probably not widely known. In any case, the teacher had to sit down and verify their hypotheses. Abdullah, Ceren and Kevin proposed that

$$a_n = a_1 + \sum_{i=1}^{k} \frac{\lambda_i}{i!} \prod_{j=1}^{i} (n-j)$$

where the coefficient λ_i denotes the first term in the i - th sequence of differences. It should be noted that

$$\frac{1}{i!} \prod_{j=1}^{i} (n-j) = \binom{n-1}{i}$$

where any binomial coefficient is zero if $i > n-1$. Furthermore, let $a_1 = \lambda_0$ and the students’ statement can be compacted into

$$a_n = \sum_{i=0}^{k} \lambda_i \binom{n-1}{i} .$$

If we now build the first sequence of differences, we find that

$$d_n = a_{n+1} - a_n = \sum_{i=0}^{k} \lambda_i \binom{n}{i} - \sum_{i=0}^{k} \lambda_i \binom{n-1}{i} = \sum_{i=0}^{k} \lambda_i \left(\binom{n}{i} - \binom{n-1}{i} \right) = \sum_{i=1}^{k} \lambda_i \binom{n-1}{i-1}$$

and through renumbering the summation index as $j = i - 1$ we obtain

$$d_n = a_{n+1} - a_n = \sum_{j=0}^{k-1} \lambda_{j+1} \binom{n-1}{j} .$$

This, in turn, is the proposed formula but for an arithmetic sequence of order $k - 1$, and thus the students' formula has been proved by induction. It may also be noted that the k - th sequence of differences is the sequence of constants, which only consists of λ_k.

It goes without saying that this proof, especially in its most general form, was not presented in class. While this would have been beyond the scope of the students, the presentation of select material produced by the students themselves allowed the teacher to draw attention to core ideas relating to binomial coefficients and their sums, which already shined through in the students' table.[5]

Second use: the continuation of the lesson takes an unplanned turn

With the help of the thus developed formula, students can now and on their own solve a problem that a teacher usually has to demonstrate in a row of laborious calculations or through proof by induction: finding a formula for the sum of the first n perfect squares. Based on the above tabling structures for sequences and their underlying sequences of differences, it becomes clear that the series of perfect squares is a 2nd order arithmetic sequence (highlighted in the following table). As a consequence, its partial sums build an arithmetic 3rd order sequence. In keeping with the above tabling structure, we obtain:

Term	a_1	a_2	a_3	a_4	a_5	a_6	...
Term value	1	5	14	30	55	91	...
1. seq. of differences	4	9	16	25	36	...	
2. seq. of differences	5	7	9	11	...		
3. seq. of differences	2	2	2	...			

Thus, $a_1 = 1$, $d = 4$, $e = 5$ and $f = 2$, and for the general term of this 3rd order sequence we find

$$a_n = 1 + 4 \cdot (n-1) + 5 \cdot \frac{(n-1)(n-2)}{2} + 2 \cdot \frac{(n-1)(n-2)(n-3)}{2 \cdot 3}$$

[5] The website „http://www.wias-berlin.de/people/stephan/folgen.htm" (09.10.2012) provides a link "Zahlenfolgen (number sequences)" that leads to a PDF article by Holger Stephan (Weierstraß-Institut für Angewandte Analysis und Stochastik, Berlin). Chapter 1.4 "Arithmetic sequences and Pascal's triangle" deals with the same insight that the students Abdullah, Ceren and Kevin had.

A simple term manipulation results in the well-known formula for the sum of the first n perfect squares

$$a_n = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} = \frac{n(n+1)(2n+1)}{6}$$

This now opens the flood gates for formulae for series of natural numbers with higher exponents. From the respective table for the 1st sequence of differences 8, 27, 64, 125. … (which the reader may do himself) follows that

$$\sum_{i=1}^{n} i^3 = 1 \cdot \binom{n-1}{0} + 8 \cdot \binom{n-1}{1} + 19 \cdot \binom{n-1}{2} + 18 \cdot \binom{n-1}{3} + 6 \cdot \binom{n-1}{4} = \frac{n^2(n+1)^2}{4}$$

Conclusion: Inquiry-based mathematical education (IBME) can start with elementary questions, will often take an unplanned turn, may lead to a subject-related in-depth discussion, and enables the learner to gain insight into the methods and thinking of co-students.