The quadratic equation – this is a topic to be taught in an inquiry-based way after all

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Sooner or later, every student attending a Secondary School has to deal with the quadratic equation and its formula to solve it. It is remarkable that this famous formula does not have a specific name and that it is neither associated to a mathematician nor to a certain theorem. In German language areas, the formula is also known as "Midnight Formula" – humorously alluding to the fact that you have to learn the formula by heart so that if someone called you in the middle of the night, you'd be able to recite the formula.

In the opinion of many maths teachers, the "Midnight Formula" is a topic which cannot be taught in the spirit of "self-regulated learning", "inquiry-based lessons"[1], "constructivism", "individual and cooperative problem-based learning" or "dialogic learning." These teachers claim that the formula is of such high complexity that it has to be provided and proven by the teacher himself and is, therefore, incompatible with modern teaching methods as the ones mentioned above. They doubt that it can be taught in a "self-organised framework" unless the teacher supplies at least a kind of supportive master program[2] to the students.

This article, which was published in 1995 and in 2006[3], presents an exercise that allows teachers to let their students investigate the problem field of the quadratic equation in a self-organised way. All in all, the exercises cover about four lessons. These lessons follow the pattern of dialogic learning[4]:
The teacher hands out the table attached at the bottom. In a first step, this table is the "task" the students have to deal with for the time of about one lesson (at home or at school). The term "task" means that the students not only solve the exercises, but that they are asked to write down all of their ideas and every working step in a so-called "learning journal". For this first step, it is of course not necessary that the students solve all of the 14 exercises at once. However, it is very important that they do work on this task with great care. During the following lesson, the teacher collects the journals, chooses a few interesting contributions and discusses them in class. This first phase is followed by a second one, in which the students work on the rest of the exercises (e.g. from exercise No. 6 onwards). Again, the students write down their ideas into their journals during the span of about one lesson. The journals are again collected by the teacher, who chooses some of the more beneficial excerpts and photocopies them. These photocopies are handed out to the class as a so-called "collection of autographs", which is then discussed in class.

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[1] „Inquiry-Based Science and Mathematics Education" (IBSME) is the aim of the EU-project FIBONACCI, which is conducted from 2010 to 2013 in several European countries.

[2] Such a master program (50 numbers of pages, a concept for 8 to 12 lessons) can be downloaded from the EducETH-server of the ETH: http://www.educ.ethz.ch/unt/um/mathe/aa/quadr_gleich/index (21.10.2010)

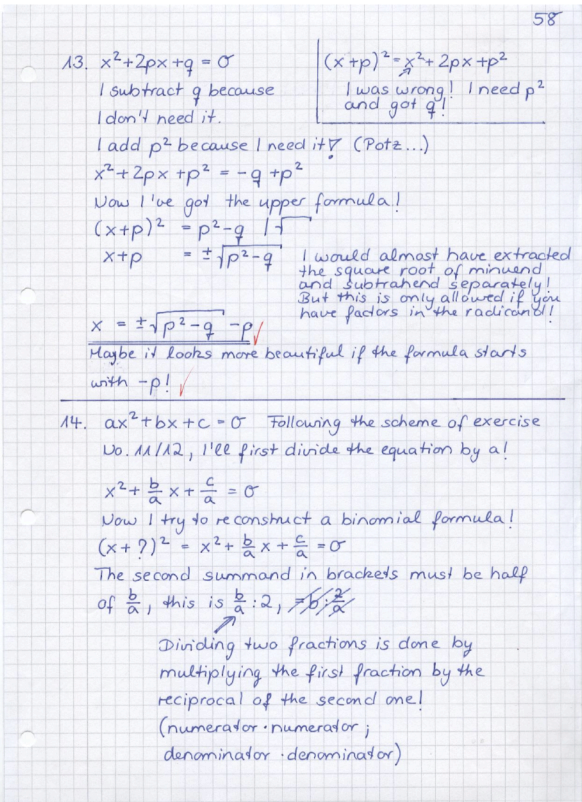
[3] Gallin, Peter; Ruf, Urs (1995): *Singuläre Schülertexte als Basis eines allgemeinbildenden Mathematikunterrichts*. In: Bieler, Heymann, Winkelmann (Hrsg.): *Mathematik allgemeinbildend unterrichten: Impulse für Lehrerbildung und Schule.* IDM-Reihe Band 21, Aulis Verlag Deubner Köln.
Gallin, Peter (2006): *Zur Auflösungsformel für die quadratische Gleichung.* In: Praxis der Mathematik in der Schule, Heft 7, Februar 2006/48 Jg., S. 45 – 46. Köln: Aulis Verlag Deubner.

[4] Ruf, Urs & Gallin, Peter (2005): *Dialogisches Lernen in Sprache und Mathematik. Austausch unter Ungleichen. Grundzüge einer interaktiven und fächerübergreifenden Didaktik* (Band 1) und *Spuren legen – Spuren lesen. Unterricht mit Kernideen und Reisetagebüchern* (Band 2). 3. überarbeitete Auflage. Seelze-Velber: Kallmeyer.

Only little instruction has to be given as an initial impulse to make the students work. This is what is so special about this task. Texts of remarkable content are delivered by students who actually manage to follow these instructions and who dare to write down all of their ideas while working on the task. These texts are worth being introduced to the other students in the class as part of a collection of autographs. This is the idea of "dialogic learning": it arises whenever students' texts are used as the point of departure for further progress in the following lessons.

The small space between the different exercises on the master copy may lead to the misunderstanding that students only have to write short answers. It is, therefore, necessary to tell the students not to write on the copy itself but into their learning journals, where there is enough space for detailed answers and ideas on their investigation of the quadratic equation. Concerning previous knowledge, the teacher of course has to make sure that the class does not know the quadratic equation yet, but that the students are familiar with square roots and know how to deal with them.
Having enough time to work on the task (it may also be finished as homework) about half the students in classes of average abilities manage to solve the last, most general exercise. The majority of students should be able to find the correct answers to exercise No. 11 or 12. Consequently, it is indeed possible to let students derive the general formula to solve quadratic equations on their own.

Vreni, a girl in my former class, had a wonderfully original idea, a so-called core idea, which lead to the technique of completing the square. With the help of this idea, she derived the general formula to solve any quadratic equation. Excerpts from her journal, which deal with exercise No. 13 and 14 are presented here:



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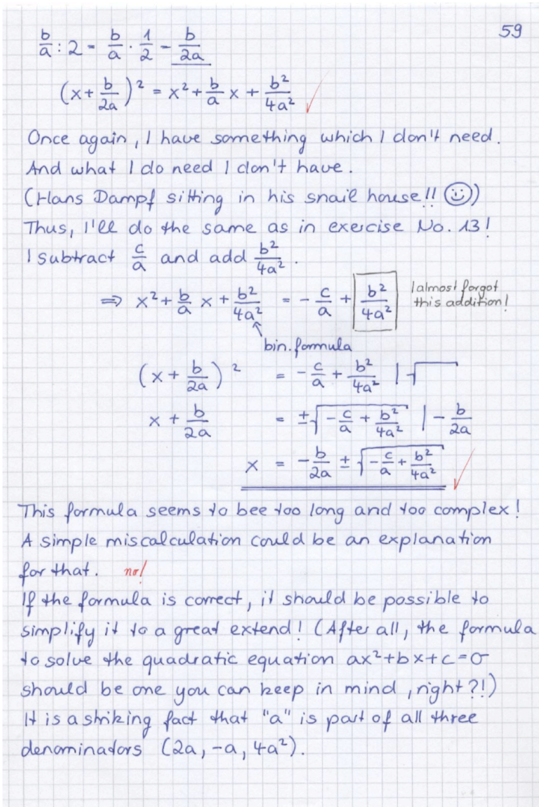
13. $x^2+2px+q=0$
I subtract q because
I don't need it.

$(x+p)^2 = x^2+2px+p^2$
I was wrong! I need p^2
and got q!

I add p^2 because I need it! (Potz...)
$x^2+2px+p^2 = -q+p^2$
Now I've got the upper formula!
$(x+p)^2 = p^2-q \quad |\sqrt{}$
$x+p = \pm\sqrt{p^2-q}$
I would almost have extracted the square root of minuend and subtrahend separately! But this is only allowed if you have factors in the radicand!
$x = \pm\sqrt{p^2-q}-p$
Maybe it looks more beautiful if the formula starts with $-p$!

14. $ax^2+bx+c=0$ Following the scheme of exercise No. 11/12, I'll first divide the equation by a!
$x^2+\frac{b}{a}x+\frac{c}{a}=0$
Now I try to reconstruct a binomial formula!
$(x+?)^2 = x^2+\frac{b}{a}x+\frac{c}{a}=0$
The second summand in brackets must be half of $\frac{b}{a}$, this is $\frac{b}{a}:2$, $\cancel{=b:\frac{2}{a}}$
Dividing two fractions is done by multiplying the first fraction by the reciprocal of the second one!
(numerator · numerator ;
denominator · denominator)



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$\frac{b}{a}:2 = \frac{b}{a}\cdot\frac{1}{2} = \frac{b}{2a}$
$(x+\frac{b}{2a})^2 = x^2+\frac{b}{a}x+\frac{b^2}{4a^2}$

Once again, I have something which I don't need.
And what I do need I don't have.
(Hans Dampf sitting in his snail house!! ☺)
Thus, I'll do the same as in exercise No. 13!
I subtract $\frac{c}{a}$ and add $\frac{b^2}{4a^2}$.
$\Rightarrow x^2+\frac{b}{a}x+\frac{b^2}{4a^2} = -\frac{c}{a}+\boxed{\frac{b^2}{4a^2}}$ I almost forgot this addition!
bin. formula
$(x+\frac{b}{2a})^2 = -\frac{c}{a}+\frac{b^2}{4a^2} \quad |\sqrt{}$
$x+\frac{b}{2a} = \pm\sqrt{-\frac{c}{a}+\frac{b^2}{4a^2}} \quad |-\frac{b}{2a}$
$x = -\frac{b}{2a}\pm\sqrt{-\frac{c}{a}+\frac{b^2}{4a^2}}$
This formula seems to bee too long and too complex! A simple miscalculation could be an explanation for that. no!
If the formula is correct, it should be possible to simplify it to a great extend! (After all, the formula to solve the quadratic equation $ax^2+bx+c=0$ should be one you can keep in mind, right?!)
It is a striking fact that "a" is part of all three denominators $(2a, -a, 4a^2)$.

Working on exercise No. 13 she wrote: "I need p^2 but (instead) I got q", and "I subtract q because I don't need it. I add p^2 because I need it! (Potz…)". "Potz…" is an interjection signifying that she has realised something novel. Not until exercise No. 14 does she let the reader know the source of her discovery: an old Swiss mockery rhyme came to her mind

called “Hans Dampf im Schnäggeloch.”[5] This rhyme surprisingly comprises the idea of “completing the square”: you have something which you don’t need, while at the same time there is a lack of something you need.
Here is the original Swiss rhyme and the attempt of an English translation:

De Hans Dampf im Schnäggeloch	Hans Dampf sitting in his snail house[*]
Hät alles was er will,	Has everything he wants to have
Und was er will,	But what he wants
Das hät er nöd,	He does not have
Und was er hät,	And what he has
Das will er nöd,	He does not want
De Hans Dampf im Schnäggeloch	Hans Dampf sitting in his snail house
Hät alles was er will.	Has everything he wants to have.

It is obvious that Vreni’s core idea must be kept secret as long as her classmates still work on the task. As a consequence, working with this task in class will lead to more and more unexpected and quite different discoveries made by Vreni’s classmates – as long as they have enough time to write down their own associations and ideas into their journals. In this case, Vreni’s journal excerpt was handed out to the others as an autograph and was discussed in class. At the end of her notes, Vreni doubts she’s right because her formula seems to be too complicated. Her doubt and the teacher’s comment on it (a plain “no!”) were elaborately discussed in the following lesson. The shape of the “official formula,” as it can be found in formularies

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

can be derived from Vreni’s formula in only a few steps. At the same time, the students who didn’t manage to solve exercise No. 14 are informed about the derivation of this most general official formula. Students are generally able to follow a derivation easily when it is demonstrated by means of an authentic text written by a classmate rather than being presented by the teacher. Moreover, every student has dealt with the problem at great length. For this reason, all of them are able to ask explicit questions and they know their “personal pitfalls”, which can be clarified during class discussion. The difference to most ordinary classroom settings is that the students receive answers to questions they really do have and not answers to questions they hadn’t even thought of. As a consequence, self-organised and individual research on a topic culminates in cooperative learning, where students talk to other students about the class topic.

The above example shows that even well-established topics of the maths-curriculum lend themselves to be taught in an inquiry-based way, as long as they are carefully prepared by the teacher. By doing research on a question with the help of a suitable task, every single student can approach the question and some students may even answer it. In any case, the dialogue among the students will transform them into a “research community” in which every single student can gain a deeper insight into the topic. This is due to an atmosphere of cooperation in which the group is meant to achieve rather than having the students all work on their own.

[5] Suter, Robert (1915): *Am Brünneli, am Brünneli, Schweizer Kinderreime.* Sauerländer Verlag Aarau.

[*] Annotation of the translator: There are several translations of the word “Schnäggeloch“. Besides the meaning of “snail house” some people claim that it means “mosquito-hole“ (“Schnägge” as a derivation of the German word for mosquito: “Schnake”), others are of the opinion that it refers to an Austrian cave called “Schneckenlochhöhle”. I’ve chosen the translation of “snail house” because it fits best to “Hans Dampf” who is unhappily sitting in his snail house complaining about what he has (not) and what he actually wants.

The cooperative total achievement of the group can, therefore, surpass the merit of a single student working on his own.
Teachers willing to teach in an inquiry-based way often think that they have to find problems which are extraordinarily inventive and "nice", and which are at most also related to everyday life. The example discussed above shows that it is much more important that students get personally involved in the problem field, that they dare to state their feelings and thoughts, even if they are incorrect, and that their contributions are taken seriously by their classmates as well as by the teacher. This is exactly what so-called "dialogic lessons"[6] aim at.
Accordingly, the students' notes in their journals represent not only "draft papers" for their studies but are equally subject to the performance measurement just as conventional tests are.

[6] For further information on „dialogue-based teaching" see reference No. 4 or recent publications by Urs Ruf, Stefan Keller, Felix Winter (editors) (2008): *Besser lernen im Dialog. Dialogisches Lernen in der Unterrichtspraxis*. Seelze-Velber: Klett und Kallmeyer.

The formula to solve the quadratic equation $ax^2 + bx + c = 0$

Task: Solve each of the following equations. As a general rule, each equation has got two (!) solutions. Write down all of your thoughts and, above all, write down how the question at hand differs from the previous one.
Make sure not to expand the term in exercise No. 5 and use No. 5 to solve No. 6.
Where you have square roots, make sure that
a) there is **no square root in the denominator**, i.e. rationalise the denominator.
b) you **simplify each square** root as far as possible.

1. $x^2 = 4$
2. $x^2 - 3 = 0$
3. $2x^2 - 1 = 0$
4. $x^2 = 6$
5. $(x+2)^2 = 6$
6. $x^2 - 6x + 9 = \frac{25}{4}$
7. $x^2 - 6x \quad = 31$
8. $x^2 + 4x \quad = -\frac{7}{4}$
9. $x^2 - \frac{2}{3}x \quad = -\frac{1}{9}$
10. $x^2 - 3x \quad = -\frac{25}{4}$
11. $2x^2 + 4x - 7 = 0$
12. $\frac{1}{6}x^2 - \frac{1}{4}x - \frac{1}{6} = 0$
13. $x^2 + 2px + q = 0$
14. $ax^2 + bx + c = 0$